MATH 3235 Probability Theory

10/25/22

$X \quad Y \qquad f(x,y)$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$\forall y \quad f_{X|Y}(x|y)$ is p.d.f. on x

$$\int_{\mathbb{R}} f_{X|Y}(x|y)\,dx = \frac{\int_{\mathbb{R}} f(x,y)\,dx}{f_Y(y)} = 1$$

$\mathbb{P}(A|B)$ is a probability as a function of A.

$$\mathbb{E}(X \mid Y=y) = \int_{\mathbb{R}} f_{X|Y}(x|y)\, x \, dx$$

Conditional Expectation of X given $Y=y$.

$$\mathbb{E}(X) = \int_{\mathbb{R}} \mathbb{E}(X \mid Y=y) f_Y(y)\, dy =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(x,y)}{f_Y(y)}\, x\, dx \; f_Y(y)\, dy$$

$$= \int_{\mathbb{R}^2} f(x,y)\, x \, dx\, dy$$

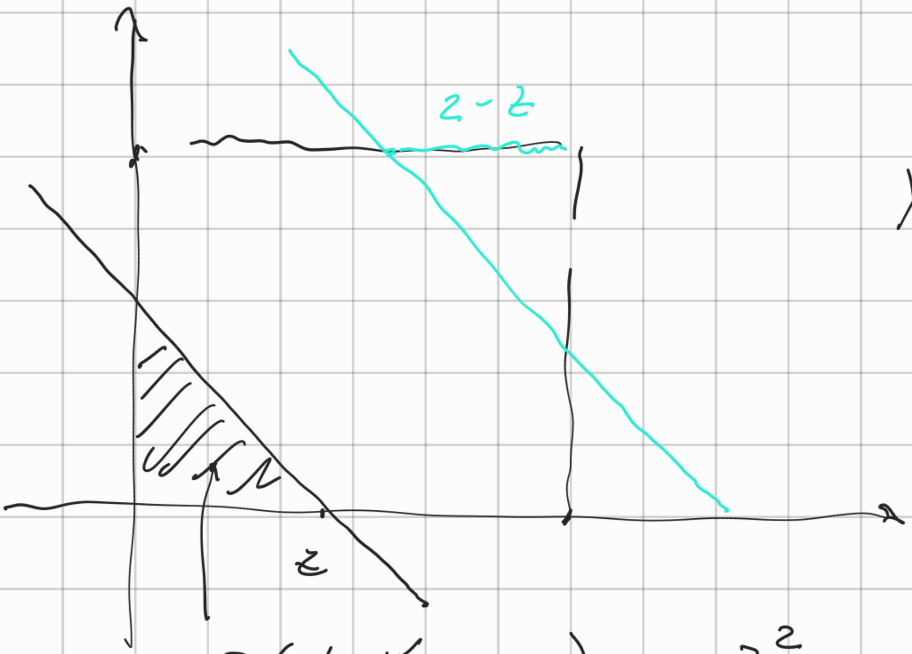
X Y uniform in $[0,1]$

$Z = X+Y$ $\qquad X \perp\!\!\!\perp Y$.

The p.d.f. of Z?

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{\mathbb{R}} f_X(x)\, f_Y(z-x)\, dx$$



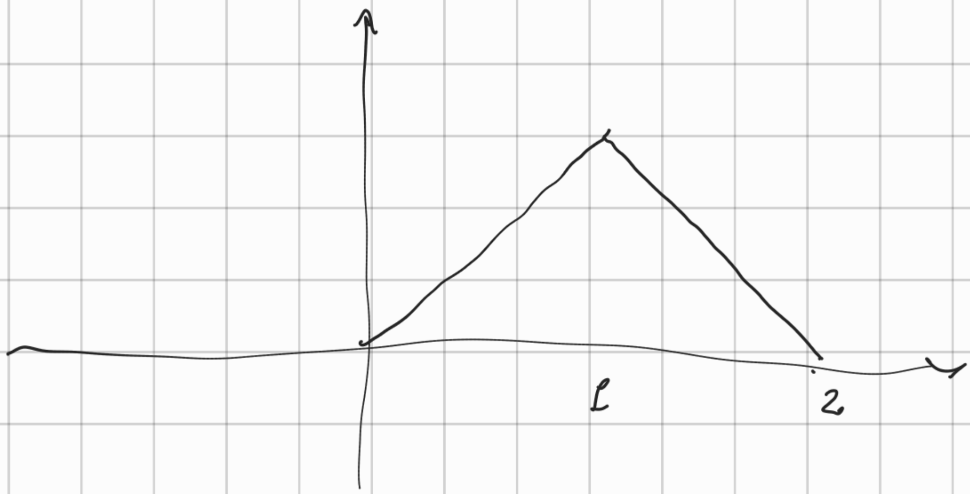
$X + Y \leq z$

$z < 1$

$$\mathbb{P}(X+Y < z) = \frac{z^2}{2} \qquad 0 \leq z \leq 1$$

$$\mathbb{P}(X+Y \leq z) = 1 - \frac{(2-z)^2}{2} \qquad 1 \leq z \leq 2$$

$$f_Z(z) = \begin{cases} z & 0 \leq z \leq 1 \\ 2 - z & 1 \leq z \leq 2 \end{cases}$$



$X_1 + X_2 + X_3 =$

$\sum_{i=1}^{N} X_N$ where N may be 40 or 100.

$N(\mu, \sigma^2) \sim X$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\mathbb{E}(X) = \mu \qquad Var(X) = \sigma^2$

If $X \sim N(0, 1)$ Then

$Y = \sigma X + \mu \sim N(\mu, \sigma^2)$

$\mathbb{E}(Y) = \sigma \mathbb{E}(X) + \mu = \mu$

$Var(Y) = \sigma^2 Var(X) = \sigma^2$

If $X \sim N(\mu, \sigma^2)$ Than

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

$$\mathbb{P}(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx = \Phi(z)$$

$$X \sim N(\mu, \sigma^2)$$

$$\mathbb{P}(X \le x) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) =$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\mathbb{P}(x_1 \le X \le x_2) = \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right)$$

Find z such That

$$\mathbb{P}(Z \ge z) = 0.05$$

$$\mathbb{P}(Z \le z) = 0.95$$

$$z = \Phi^{-1}(0.95) = 1.36 \ (?)$$

$z_{0.05}$ critical value.

X_1 and X_2 are Normal Standard independent

$Z = X_1 + X_2$

Z is also Normal

$\mathbb{E}(Z) = 0 \qquad V(Z) = 2$

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{(z-x)^2}{2}} dx$$

$$= \frac{1}{2\pi} \int e^{-\frac{x^2}{2} - \frac{z^2}{2} + xz - \frac{x^2}{2}} dx$$

$$-x^2 + xz - \frac{z^2}{2} =$$

$$-x^2 + xz - \frac{z^2}{4} - \frac{z^2}{4} =$$

$$= -\left(x - \frac{z}{2}\right)^2 - \frac{z^2}{4}$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{\mathbb{R}} e^{-\left(x-\frac{z}{2}\right)^2} dx$$

$y = x - \frac{z}{2}$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{\mathbb{R}} e^{-y^2} dy =$$

$$= \frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$$

$\sigma^2 = 2$

X temperature measurement.

$N(\mu, \sigma^2)$ μ is true value

σ^2

$X_1, X_2 \ldots X_N$ many times.

$\bar{X} = \frac{1}{N} \sum_i X_i$ $\bar{X} \simeq N\left(\mu, \frac{\sigma^2}{N}\right)$

find δ sucht that $(x - \delta, x + \delta) \ni \mu$.

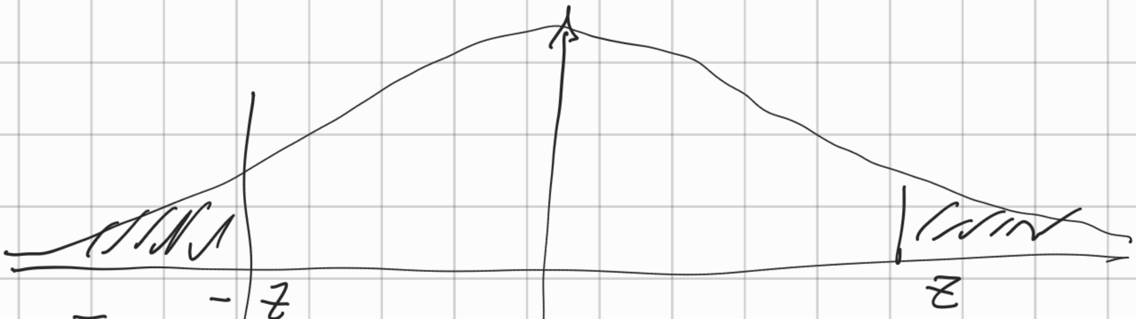
with prob 0.95.

If X is $N(\mu, \sigma^2)$

$$\mathbb{P}(\mu - \delta \le X \le \mu + \delta) = 0.95$$

$$\mathbb{P}\left(-\frac{\delta}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{\delta}{\sigma}\right) =$$

$$\Phi\left(\frac{\delta}{\sigma}\right) - \Phi\left(-\frac{\delta}{\sigma}\right) = 0.95$$



$$\Phi(-z) = 1 - \Phi(z)$$

$$2\,\Phi\left(\frac{\delta}{\sigma}\right) - 1 = 0.95$$

$$\Phi\left(\frac{\delta}{\sigma}\right) = 0.025$$

$$\delta = z_{0.025}\,\sigma$$

1 measurement x

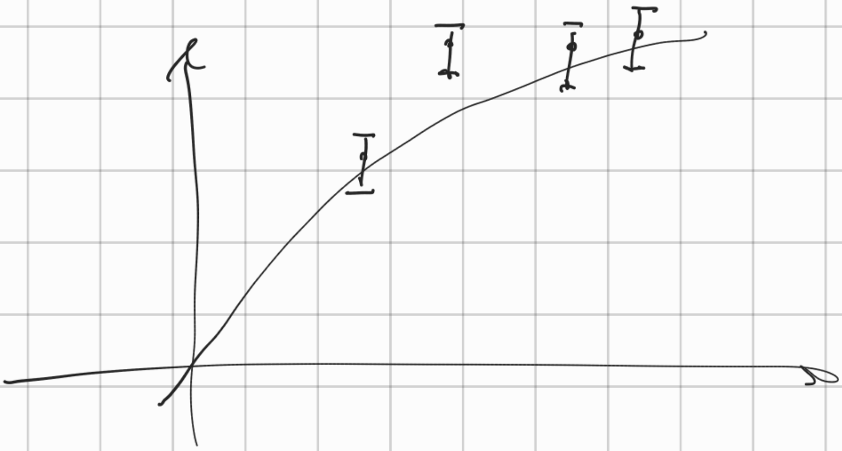
with 95%

$$\mu \in [x - z_{0.025}\sigma, \; x + z_{0.025}\sigma]$$

N measurement $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$

$$\mu \in [x - z_{0.025}\frac{\sigma}{\sqrt{N}}, \; x + z_{0.025}\frac{\sigma}{\sqrt{N}}]$$

95%



$$f(x,y) = \begin{cases} e^{-y} & 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

X, Y

Marginal y

$$\int_{\mathbb{R}} f(x,y)\,dx = \int_0^y e^{-y}\,dx = y e^{-y}$$

$$\int_{\mathbb{R}} f(x,y)\,dy = \int_x^\infty e^{-y}\,dy = e^{-x}$$

X is exponentical.

$X_2 = Y - X$.

X X_2 are both exponoutical

and independent

$$e^{-y} \quad 0 \le x \le y \implies e^{-(x + X_2)} \quad \begin{matrix} x > 0 \\ x_2 > 0 \end{matrix}$$

$$f_{X|Y}(x|y) = \frac{1}{x} \qquad x \leq y$$

Generate Two in dip. exp.

Generate a number with dist.

$y e^{-y}$.